A Novel FDTD Formulation for Dispersive Media

Shaobin Liu, Naichang Yuan, and Jinjun Mo

***Abstract*—A novel FDTD formulation for dispersive media called piecewise linear JE recursive convolution (PLJERC) finite-different time-domain (FDTD) method is derived using the piecewise linear approximation and the recursive convolution relationship between the current density J and the electric field $\mathbf{E}$. The high accuracy and efficiency of the PLJERC method is confirmed by computing the reflection coefficients of electromagnetic wave through a collision plasma slab in one dimension.**

***Index Terms*—FDTD methods, plasma, recursive convolution.**

I. Introduction

THE finite-different time-domain method has been widely used to simulate the transient solutions of electromagnetic wave propagation in various media including dispersive media. Over the last decade, there have been numerous investigations of FDTD dispersive media formulations. These include the recursive convolution (RC) methods [1]–[5], the auxiliary differential equation (ADE) methods [6]–[10], frequency-dependent Z transform methods [11], [12], JE convolution (JEC) method [13], piecewise linear recursive convolution (PLRC) method [14] and direct integration (DI) methods [15]–[17].

In this letter, we propose a novel method, i.e., the piecewise linear JE recursive convolution (PLJERC) method that is derived using the piecewise linear approximation and the recursive convolution relationship between the current density and the electric field. This method had greatly improved accuracy, but had retained high speed and efficiency advantages. The high accuracy and efficiency of the PLJERC method is confirmed by computing the reflection coefficients of electromagnetic wave through a collision plasma slab in one dimension.

II. Methodology

Considering an isotropic collision plasma medium. The Maxwell's equations and constitutive relation for an unmagnetized collision plasma are given by

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \tag{1}$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{2}$$

$$\frac{\partial \mathbf{u}_e}{\partial t} = -\frac{e}{m}\mathbf{E} - \nu \mathbf{u}_e \tag{3}$$

$$\mathbf{J} = -e n_e \mathbf{u}_e \tag{4}$$

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The authors are with the Institute of Electronic Science and Engineering, National University of Defense Technology, Changshai 410073, China (e-mail: liushaobin273@sohu.com).

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here, $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic intensity, $\mathbf{J}$ the polarization current density, ε_0 the permittivity of free space, μ_0 the permeability of free space, n_e is the electron concentration, ν the electron collision frequency, and e and m are the electric charge and mass of an electron, respectively.

For a time-harmonic dependence, the following frequency-domain relationship between $\mathbf{E}$ and $\mathbf{J}$ can be derived from (3) and (4),

$$\mathbf{J}(\omega) = \varepsilon_0 \frac{\omega_p^2}{j\omega + \nu}\mathbf{E}(\omega) = \sigma(\omega)\mathbf{E}(\omega) \tag{5}$$

where $\omega_p = \sqrt{n_e e^2 / m\varepsilon_0}$ is the plasma frequency, and

$$\sigma(\omega) = \varepsilon_0 \frac{\omega_p^2}{j\omega + \nu} \tag{6}$$

Taking the inverse transform of (5) and (6), we obtain by use the convolution integral

$$\mathbf{J}(t) = \int_0^t \mathbf{E}(t-\tau)\sigma(\tau)d\tau \tag{7}$$

$$\sigma(\tau) = \varepsilon_0 \omega_p^2 \exp(-\nu\tau)U(\tau) \tag{8}$$

where $U(\tau)$ is the unit step function.

Using Yee's notation, we let $t = n\Delta t$ in (7), and the each vector component of $\mathbf{J}$ and $\mathbf{E}$ can be written as

$$J_i(n\Delta t) = J_i^n = \int_0^{n\Delta t} E_i(n\Delta t - \tau)\sigma(\tau)d\tau \tag{9}$$

where $i = x, y, z$.

We now introduce the D. F. Kelley *et al.*'s piecewise linear approximation [14]. The cornerstone of this method is that the electric field is assumed to have a piecewise linear functional dependence over Δt. The each vector component of electric field that appears in the convolution in (9) can be expressed as (As shown in Fig. 1(b) of [14])

$$E_i(n\Delta t - \tau) = E_i^{n-m} + \frac{E_i^{n-m-1} - E_i^{n-m}}{\Delta t}(\tau - m\Delta t) \tag{10}$$

Substitution of (10) and (8) into (9), after some manipulation the each component of J at n time step can be written as

$$J_i^n = \sum_{m=0}^{n-1} \left[E_i^{n-m}\sigma^m + \left(E_i^{n-m-1} - E_i^{n-m}\right)\xi^m\right] \tag{11}$$

where $i = x, y, z$ and

$$\begin{aligned}\sigma^m &= \int_{m\Delta t}^{(m+1)\Delta t} \sigma(\tau)d\tau \\ &= \frac{\varepsilon_0 \omega_p^2}{\nu}\left[1 - \exp(-\nu\Delta t)\right]\exp(-m\nu\Delta t)\end{aligned} \tag{12}$$

$$\xi^m = \frac{1}{\Delta t}\int_{m\Delta t}^{(m+1)\Delta t}(\tau - m\Delta t)\sigma(\tau)d\tau = \frac{\varepsilon_0\omega_p^2}{\nu^2\Delta t}\left[1-(1+\nu\Delta t)\exp(-\nu\Delta t)\right]\exp(-m\nu\Delta t). \quad (13)$$

And the each component of J at next time step can be written as

$$J_i^{n+1} = \sum_{m=0}^{n}\left[E_i^{n+1-m}\sigma^m + \left(E_i^{n-m} - E_i^{n+1-m}\right)\xi^m\right]. \quad (14)$$

From (11) and (14), we obtain

$$J_i^{n+1} + J_i^n = \left(\sigma^0 - \xi^0\right)E_i^{n+1} + \xi^0 E_i^n + \sum_{m=0}^{n-1}\left[E_i^{n-m}\left(\sigma^m + \sigma^{m+1}\right) + \left(E_i^{n-m-1} - E_i^{n-m}\right)\left(\xi^m + \xi^{m+1}\right)\right]. \quad (15)$$

The summation in (15) is assigned to a recursion accumulator ψ_i^n. Thus

$$\psi_i^n = \sum_{m=0}^{n-1}\left[E_i^{n-m}\left(\sigma^m + \sigma^{m+1}\right) + \left(E_i^{n-m-1} - E_i^{n-m}\right)\left(\xi^m + \xi^{m+1}\right)\right]. \quad (16)$$

From (12)–(13), we note that

$$\sigma^m = \exp(-\nu\Delta t)\sigma^{m-1} \quad (17)$$

$$\xi^m = \exp(-\nu\Delta t)\xi^{m-1}. \quad (18)$$

The update equation of the recursive accumulator is

$$\psi_i^n = \left(\sigma^0 + \sigma^1 - \xi^0 - \xi^1\right)E_i^n + \left(\xi^0 + \xi^1\right)E_i^{n-1} + \exp(-\nu\Delta t)\psi_i^{n-1}. \quad (19)$$

By using central difference formulas, we generate the following y component FDTD equation for (1) as

$$(\nabla \times \mathbf{H})_y^{n+1/2} = \varepsilon_0\frac{E_y^{n+1} - E_y^n}{\Delta t} + \frac{J_y^{n+1} + J_y^n}{2}. \quad (20)$$

After substituting the (15) into the (20), the general form of the FDTD update equation for E_y becomes

$$E_y^{n+1} = \frac{1}{1 + \frac{\Delta t}{2\varepsilon_0}\left(\sigma^0 - \xi^0\right)}\left[\left(1 - \frac{\Delta t}{2\varepsilon_0}\xi^0\right)E_y^n + \frac{\Delta t}{\varepsilon_0}(\nabla \times \mathbf{H})_y^{n+1/2} - \frac{\Delta t}{2\varepsilon_0}\psi_y^n\right]. \quad (21)$$

The equations for other component of electric field are similar.

It appears from (19) and (21) that the PLJERC requires one more real storage in addition to the normal FDTD storage of the electromagnetic field components.

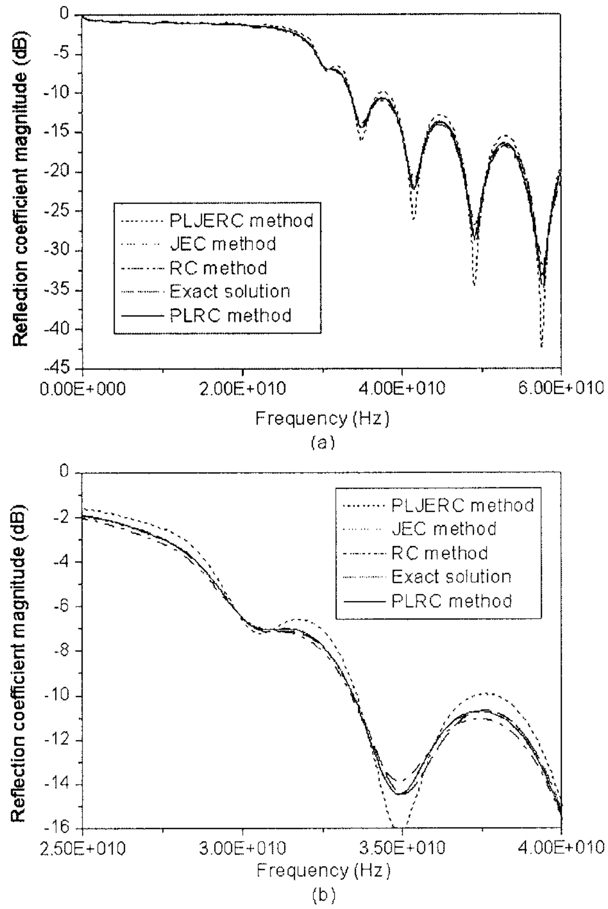

Fig. 1. Reflection coefficient magnitude computed using PLJERC, JEC, RC and PLRC methods compared with the analytical solution.

III. Verification and Accuracy

To demonstrate the aforementioned new PLJEC FDTD formulation for plasmas, we compute the reflection coefficients of electromagnetic wave through an unmagnetized collision plasma slab with a thickness of 1.5 cm. The incident wave used in the simulation is a Gaussian-derivative pulsed plane wave whose frequency spectrum peaks at 50 GHz and is 10 dB down from the peak at 100 GHz.The incident wave normally incident on plasma slab. The computational domain is 6 cm long and the plasma slab is defined by the region [2.25, 3.75] cm. For FDTD parameters, the spatial discretization, used in the simulations is 75 μm and the time step is 0.125 ps. So, the computational domain is subdivided into 800 cells. The plasma slab occupies cells 300 to 500, free space from 0 to 300 and 500 to 800. The plasma parameters were

$$\omega_p = 2\pi^* 28.7^* 10^9 \text{ rad/s} \quad (22)$$

$$\nu = 20^* 10^9 \text{ rad/s}. \quad (23)$$

Fig. 1 shows the magnitudes of the reflection coefficients computed using the PLJERC method, JEC method, RC method and PLRC method with those of the analytical solution. The reflection coefficients are calculated by computing the FFT of the time history of the reflected pulse at the interface and the FFT of the time history of the incident pulse for PLJERC, JEC, RC, and PLRC methods. For the analytic solutions, we follow the method found in Ginzburg [18]. The agreement shown in Fig. 1 between the PLJERC, PLRC methods and analytical solution are excellent and indicates a greatly improvement over both JEC and RC methods. The curve also shows that the PLJERC method can greatly improved the accuracy over the original RC method and JEC method; and minor improved accuracy over that of the PLRC method.

IV. Conclusion

The PLJERC finite-different time-domain is applied to plasma media. Numerical simulations the reflection coefficients through a plasma layer illustrate the technique. The results show that the PLJERC-FDTD method has improved the accuracy over the original RC method, JEC method and PLRC method. Meanwhile the method retains excellent computational efficiency.

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